Short Papers

Procedure for Direct Calculation of Characteristic Admittance Matrix of Coupled Transmission Lines

Karl Reiss and Olgierd A. Palusinski

Abstract— **Physical design of contemporary electronic circuits and sytems involves their analysis with interconnections modeled as transmission lines. Algorithms proposed in the literature for calculation of characteristic admittance matrix are based on eigenanalysis and have inherent ambiguities associated with this method when multiple eigenvalues occur. In this paper, the underlying theory and details of a new algorithm developed for unconditionally unique results are given.**

I. Introduction

In selecting line models for advanced microelectronic device simulation there is a trade-off between accuracy and efficiency [1]. The signal transmission is successfully modeled by lossless transmission lines which provide upper bounds for crosstalk. Delay analysis based on this line type underestimates the delay values in comparison to lossy lines but still provides worst case results for coupling noise and other performance parameters [2].

The characteristic admittance matrix $[Y_0]$ of a line system is a most important quantity for a designer. Characteristic impedance determines the levels of switching currents in the I/O drivers and thus influences the switching noise which is one of the most decisive parasitic effects in high performance electronic systems. Total transmission delay is also determined by the characteristic impedance. Ho [4] has given a detailed description of the concepts, mathematical relationships and algorithms for computations involving coupled transmission lines.

In this paper a new algorithm for computing the characteristic admittance matrix or its inverse, the characteristic impedance matrix is described. The matrix is computed directly without involving eigenanalysis thus avoiding any ambiguities that could arise in case of multiple eigenvalues. Per unit length time delays follow from the roots of the characteristic polynomial. This can be considered part of the eigenanalysis; but it is unique. It shoud be pointed out that eigenanalysis is not unique if multiple eigenvalues occur. The nonuniqueness and associated problems are discussed in the following section.

II. Problem Statement

The matrices of capacitances and inductances per unit length of line, $[C]$ and $[L]$, form the unique bridge between the physical and the electrical characteristics of the lossless line system. These matrices link the voltages and currents of the signals travelling along the line system. The respective vectors, expressed by $[V]$ and $[I]$, with $[V] = [V_1, V_2, \ldots, V_n]$, etc., are related by the equations

$$\begin{aligned}\frac{\partial}{\partial x}[V] &= -[L]\frac{\partial}{\partial t}[I]\\ \frac{\partial}{\partial x}[I] &= -[C]\frac{\partial}{\partial t}[V].\end{aligned} \tag{1}$$

The line system extends in x-direction. A reference electrode is assumed to be present but is not explicitely counted.

In order to minimize or eliminate reflections, the termination network of a line system ought to approximate or match the characteristic admittance matrix given by [4]

$$[Y_0] = [L]^{-1}[LC]^{\frac{1}{2}}. \tag{2}$$

The key computation is that of the square root of the matrix $[LC]$.

In classical analysis the product of inductance and capacitance matrices, $[LC] = [L][C]$ is diagonalized with the matrix $[P]$ such that

$$[P]^{-1}[LC][P] = [\tau_i^2] \tag{3}$$

where the right hand side is a diagonal matrix of eigenvalues τ_i^2. The columns of the matrix $[P]$ are formed by the eigenvectors.

Any signal traveling on a system of transmission lines appears as a linear superposition of (eigen)modes which travel at their respective velocities, $1/\tau_i$.

The square root of the matrix $[LC]$ is then defined using the results obtained from the eigenvalue problem

$$[LC]^{\frac{1}{2}} = [P][\tau_i][P]^{-1}. \tag{4}$$

The direct computation of square root of matrix $[LC]$ proposed in this paper is straightforward and does not involve eigenanalysis. Eigenanalysis becomes complicated when $[LC]$ matrix is asymmetric, which is a general case ($[LC]$ is symmetric in special cases only). In addition the eigenanalysis may involve some ambiguities discussed below.

The eigenvector matrix $[P]$ is unique only in the case of distinct eigenvalues. But an eigenvalue of multiplicity $m (m > 1)$ defines an m-dimensional subspace with any vector of this subspace as an eigenvector. Even if eigenvectors are submitted to orthonormalization, degrees of freedom are not completely eliminated because it is possible to generate different eigenvector matrices for the same interconnect system. If for some reason different eigenvector matrices (vis. $[P] \neq [Q]$) and their inversions are stored in a database an error resulting from using $[P]$ and $[Q]^{-1}$ in the formula (4) will occur. This error will not be detected by the system and a user will not be warned. Therefore, care must be taken when using eigenanalysis in the case of multiple eigenvalues.

In the past no attention was focused on this fact, because mainly two line systems were treated for determining the estimates of crosstalk. If multiple (double) eigenvalues occur in this case, the whole vectorspace becomes an eigenspace and eigenanalysis is unnecessary. For line systems of more than $n = 2$ conductors, multiplicity of eigenvalues was not considered in the literature.

In this short paper a more efficient algorithm for direct computation of the square root of the $[LC]$-matrix is proposed. This algorithm is more general than the one given in the cited literature because it avoids eigenanalysis and ambiguities associated with it and is applicable to other matrices as well.

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III. Mathematical Formulation

Within the scope of this article we consider only homogeneous line systems. We anticipate all materials used as conductors or dielectrics to be linear and instantaneous (local in time). This does not exclude inhomogeneous and anisotropic materials (both terms are considered within the cross section perpendicular to the conductors) but results in constant matrices $[L]$ and $[C]$. The passivity and reciprocity of the materials makes both matrices positive definite and symmetrical $[C]^T = [C], [L]^T = [L]$. The dimensions of all matrices are $n \times n$ and those of the vectors are also n unless specified otherwise. The upper index T denotes transposition and for simplicity we use $[A]$ for the matrix product $[L][C]$. Despite the symmetry of its factor matrices, the matrix $[A]$ lacks symmetry because $[L]$ and $[C]$ in general do not commute. But due to it's composition and the properties of the factor matrices specified above, a theorem of matrix theory applies [5] stating diagonalizability and proofing positive definiteness of $[A]$. Both properties are used in building the new algotithm.

The square root of the matrix $[A]$ which is known to exist from (4) but is still considered unknown, is denoted $[R]$ such that

$$[R] \cdot [R] = [A]. \tag{5}$$

The same (4) incorporates the positive definiteness of $[R]$ as well because all τ_i are positive. It should be noted here that there is no easy way to express the square root of a matrix product by the square root of its factor matrices $[LC]^{1/2} \neq [L]^{1/2}[C]^{1/2}$.

The characteristic admittance matrix $[Y_0]$ and the characteristic impedance matrix are both symmetrical and inverse to each other

$$[Y_0]^T = [Y_0], \quad [Y_0]^{-1} = [Z_0], \quad [Z_0]^T = [Z_0]. \tag{6}$$

IV. Solution Method

The proposed solution method exploits the root defining (5) directly. The root matrix $[R]$ is considered to contain n^2 unknown matrix elements r_{ik}. Equation (5) is rewritten explicitly

$$\sum_{l=1}^{n} r_{il} \cdot r_{lk} = a_{ik}; \quad 1 \leq i \leq n, \quad 1 \leq k \leq n. \tag{7}$$

The equations for the unknowns r_{ik} are nonlinear, specifically they are of quadratic nature, and their solution can be obtained by multidimensional Newton iteration. For this purpose the system of equations is written in the form

$$f_{ik} = \sum_{l=1}^{n} r_{il} \cdot r_{lk} - a_{ik} = 0 \tag{8}$$

because Newton's method searches zeros of a set of functions f_{ik}. The set of functions is also considered to be a matrix $[F]$.

The associated Jacobian tensor $[J]$ is

$$j_{lm}^{ik} = \frac{\partial f_{ik}}{\partial r_{lm}} = r_{il} \cdot \delta_{km} + r_{mk} \cdot \delta_{il} \tag{9}$$

where the symbol δ_{ij} represents the Kronecker symbol.

The unknowns and equations are specified by two indices, and thus the problem appears as a matrix equation containing a tensor of rank 4 as the Jacobian. The tensor representation would be unusual for computer implementation because there are no standard numerical packages for such a formulation. Therefore the index pair i, k is mapped into one index α only by $\alpha = (k-1) \cdot n + i$ within the range $1 \leq \alpha \leq n^2$. The same mapping applies to the pair l, m. Greek indices running from 1 to n^2 replace index pairs from previously introduced indices i, k, l, m running only from 1 to n.

Within this new indexing the matrices $[R]$ and $[F]$ are considered as vectors of dimension n^2 and the Jacobian tensor $[J]$ becomes a square matrix of dimensions $n^2 \times n^2$. The resulting Newton formula is

$$[R]^{(\nu+1)} = [R]^{(\nu)} - [J^{-1}]^{(\nu)} \cdot [F]^{(\nu)} \tag{10}$$

where the upper index (ν) is the iteration counter.

An estimate for the root matrix is possible due to the positive definiteness of the matrix $[A]$ which guarantees that this matrix is diagonally dominant. This property transfers to the root matrix $[R]$. A good approximation within the attraction region of the root matrix solution in Newton's algorithm is the diagonal matrix composed of the roots from the diagonal elements only of matrix $[A]$

$$r_{ik}^{(0)} = \sqrt{a_{ii}} \cdot \delta_{ik}. \tag{11}$$

This procedure is used to compute the square root of the matrix $[LC]$.

SQUARE ROOT OF A REAL POSITIVE DEFINITE MATRIX				
INPUT ORIGINAL 2 X 2 MATRIX BY ROWS				
A11	A12	52.6050	4.5624	
A21	A22	5.9579	49.9503	
ITER. COUNT	R11	R12	R21	R22
1	7.25293	.31859	.41604	7.06755
2	7.24377	.31901	.41658	7.05815
3	7.24376	.31901	.41658	7.05814
4	7.24376	.31901	.41658	7.05814
5	7.24376	.31901	.41658	7.05814
6	7.24376	.31901	.41658	7.05814

Fig. 1. Root calculation example, $[A]$ given, $[R]$ root.

V. Example

Transmission line data from typical off chip interconnections in simple linear circuit environment is considered. The given matrices $[L]$ and $[C]$ are for a coupled two conductor plus ground line system [1]. Although the algoritm was programmed and tested for unlimited number of lines n, the two line example given here demonstrates the convergence properties by a compact printout and is itself of considerable practical importance. The capacitance and inductance matrices are

$$[C] = \begin{pmatrix} 126.546 & -20.644 \\ -20.644 & 106.104 \end{pmatrix} pF/m, \qquad [L] = \begin{pmatrix} 436.57 & 127.94 \\ 127.94 & 495.66 \end{pmatrix} nH/m. \tag{12}$$

A factor 10^{-18} is extracted from the matrix product $[LC]$ for better representation of the printout results. Fig. 1 shows the progress of iteration and contains the numerical values.

The resulting root matrix is multiplied with 10^{-9} (the root of the extracted factor). The characteristic admittance matrix $[Y_0]$ computed using (2) is as follows

$$[Y_0] = [L]^{-1}[LC]^{\frac{1}{2}} = \begin{pmatrix} 17.684 & -3.727 \\ -3.727 & 15.204 \end{pmatrix} mS. \tag{13}$$

VI. Numerical Considerations

The Newton algorithm given in Section III on mathematical problem formulation does not require any inversion of the Jacobian $[J]^{(\nu)}$. It is only necessary to calculate the correction term, a vector $[S]^{(\nu)}$ from $[J]^{(\nu)}[S]^{(\nu)} = [F]^{(\nu)}$ which leads to (10) rewritten as $[R]^{(\nu+1)} = [R]^{(\nu)} - [S]^{(\nu)}$.

There is a slowdown of the calculations rather than a speedup and gain of accuracy with more sophisticated solvers for systems of simultaneous linear equations. In each iteration the equation must be solved only once before being updated for the next step.

Therefore $L - U$ factorization would be a wasted effort. Pivoting is not necessary, because the Jacobian $[J]$ is and remains diagonally dominant. Iteration starts with all off diagonal elements set to 0. Later $[J]$ contains in its main diagonal always a sum of main diagonal elements r_{ii} but off diagonal only zeros or single elements r_{ik} as given in (9). Therefore simple Gauss-Jordan algorithm is superior in this application.

Due to the quadratic type of nonlinearities in defining (8) the convergence is very fast and reliable. In practical applications within line theory the root matrix is not only proven diagonally dominant but the off diagonal elements are indeed small compared with the diagonal ones. Within only two iteration steps the example from Fig. 1 shows results within 10^{-4} accuracy range.

VII. Square Root of Symmetrical Matrices

The algorithm proposed in this paper applies to an arbitrary positive definite matrix. This algorithm can be simplified if in addition the matrix is symmetric, because the root matrix will also be symmetric and therefore contains fewer unknown elements. This simplification will be applicable in general transmission line theory where some calculations involve square roots of $[L]$ or $[C]$, for example $[L]^{\frac{1}{2}}[C][L]^{\frac{1}{2}}$ or $[C]^{\frac{1}{2}}[L][C]^{\frac{1}{2}}$ respectively [3]. In some special cases the product $[LC]$ is symmetric and the simplification can also be applied.

With a slight modification of the definition of the functions f_{ik} to be zeroed by Newton's algorithm and a new index mapping for equations and variables it is possible to make the resulting Jacobian matrix $[J]$ also symmetrical and use specialized equation solvers to calculate the correction term $[S]$.

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Elimination of Spurious Solutions in the Calculation of Eigenmodes by Moment Method

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***Abstract*—In this paper the origin of nonphysical solutions obtained with Galerkin's method is described. To remove these spurious solutions a practical criterion is derived. It is shown on a patch resonator example with expansion functions satisfying edge conditions, that spurious solutions generated by the conventional approach are eliminated by the application of the proposed method.**

I. Introduction

The problem of finding the eigenmodes of a transmission line or of a cavity is a classical one. Several papers, dealing with integral methods, describe how to determine these eigenmodes by Galerkin's method [1]–[4], but the choice of expansion functions which incorporate the edge conditions seems to generate nonphysical solutions, named spurious solutions [5].

The problem of spurious solutions has been mostly developed in the context of the finite-element method: the inaccurate approximation of the zero eigenvalue and the corresponding eigenfunctions generate spurious solutions [6]. A similar result has been obtained by the authors of the present paper in the context of transverse resonance method [7]: the inaccurate approximation of the infinite eigenvalue and the corresponding eigenfunctions generate spurious solutions.

For an integral equation the origin of spurious solution is given in [8], and a criterion for their elimination is demonstrated, but this criterion gives no practical information about the choice of expansion and weighting functions.

In this paper a practical criterion for a proper choice of the expansion functions and the weighting functions is given.

II. Theory

A. Notation

The extended [9] or symbolic [10] operator concept applied to moment methods allows one to use generalized expansion functions (they are in fact linear functionals), hence we further suppose here that we use an extended operator.

Extended or symbolic operator used in functional analysis is associated to the so-called transposed operator concept rather than adjoint one, as well as the use of duality product rather than scalar product [11]. Duality product is more general than scalar product; for example it is well known that it is not mathematically possible to define a scalar product involving the "Dirac function."

The notations for spaces, transposed operator and duality product are now introduced:

Let U and V represent, respectively, the domain and the range of an operator L. The elements of U and V are functions.

Let $\tilde{U}$ represent the topological dual space of U and $\tilde{V}$ represent the topological dual space of V. The elements of $\tilde{U}$ and $\tilde{V}$ are continuous linear functionals.

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